

Metastability in the Two-Dimensional Ising Model with Free Boundary Conditions

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We investigate metastability in the two dimensional Ising model in a square with free boundary conditions at low temperatures. Starting with all spins down in a small positive magnetic field, we show that the exit from this metastable phase occurs via the nucleation of a critical droplet in one of the four corners of the system. We compute the lifetime of the metastable phase analytically in the limit $T \rightarrow 0$, $h \rightarrow 0$ and via Monte Carlo simulations at fixed values of T and h and find good agreement. This system models the effects of boundary domains in magnetic storage systems exiting from a metastable phase when a small external field is applied.

KEY WORDS: Ising model; stochastic dynamics; metastability; nucleation.

1. INTRODUCTION

Metastability is observed in many different systems close to a first order phase transition. It is a dynamical phenomenon (see, for instance, [PL]) not included in the Gibbsian formalism, which is so successful for the description of stable equilibrium states [LR, I]. The development of a full theory of metastability is desirable for its intrinsic as well as for its experimental and technological interest and it also poses challenging mathematical problems (see [CGOV, OS1, OS2]).

The metastable behavior of the nearest neighbor two dimensional Ising model for large finite volumes and small magnetic fields was analyzed in [NS1, NS2] in the zero temperature limit in the framework of the "pathwise approach" introduced in [CGOV]. In [S1] R. Schonmann,

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using arguments based on reversibility, described in detail the typical escape paths from the metastable to the stable regime. Other regimes, very interesting from the physical point of view and mathematically much more complicated (finite temperature, infinite lattice and zero magnetic field), are considered in [S2] and [SS]. The finite temperature case has also been widely studied by Monte Carlo methods, for instance in [B, BM, BS, TM1]; a complete and clear description of these numerical results can be found in [RTMS]. This case has also been studied by means of transfer-matrix and constrained-transfer-matrix methods in [PS1, PS2, GRN].

In the same asymptotic regime as in [NS1], different Ising-like hamiltonians have been considered in [KO1, KO2, NO] and the three dimensional nearest neighbor Ising model has been studied in [BC]. In [CO] the Blume-Capel model has been studied (see also [FGRN] for a study of a version of this model with weak long range interactions) and in [OS1, OS2] the problem of metastability has been investigated in a more general case.

While the above works considered systems with periodic boundary conditions, the Ising model with more general boundary conditions, such as semiperiodic, circular and octogonal domains has been studied in [RKLRN], see also [SG]. In this note we study the case of a finite lattice with free boundary conditions at low temperatures and small magnetic fields using both rigorous analysis and Monte Carlo simulations.

The case of free boundary conditions is of some technological interest: during the recording process on magnetic tapes different parts of the magnetized material consisting of fine magnetic particles are exposed to different magnetic fields, resulting in domains with different orientation of the magnetization. In order to be used as storage devices, these materials must be able to retain their magnetization for long periods in weak arbitrarily oriented magnetic fields. The study of the escape from a metastable phase in a periodic system neglects the effects of boundary domains. Such effects are modeled here by considering a lattice with free boundary conditions.

We find that although the main features of the nucleation of the stable phase are not changed, some interesting new aspects arise: in particular one can say a priori where the nucleation of the stable phase will start, that is where the critical droplet will show up.

The model and results are described in Section 2 and analyzed in Section 3 and 4. Section 5 is devoted to some brief conclusions.

2. THE MODEL AND THE RESULTS

Let us consider a two dimensional Ising model defined on a finite square $A = \{1, \dots, M\}^2 \subset \mathbb{Z}^2$ with free boundary conditions. The space of

configurations is denoted by $\Omega = \{-1, +1\}^A$ and to each configuration $\sigma \in \Omega$ is associated the energy

$$H(\sigma) = -\frac{J}{2} \sum_{\langle x, y \rangle} \sigma(x) \sigma(y) - \frac{h}{2} \sum_{x \in A} \sigma(x) \quad \sigma(x) = \pm 1 \quad (2.1)$$

where the first sum runs over all the pairs of nearest neighbors in A and $J, h > 0$. The equilibrium states are described by the Gibbs measure

$$\nu_{A, \beta}(\sigma) = \frac{e^{-\beta H(\sigma)}}{\sum_{\eta \in \Omega} e^{-\beta H(\eta)}} \quad \forall \sigma \in \Omega \quad (2.2)$$

where β is the inverse temperature.

The time evolution of the model is given by the Metropolis algorithm: given a configuration σ at time t , we pick a site x at random and then change $\sigma(x)$ to $-\sigma(x)$ with probability 1 if ΔH is ≤ 0 and $\exp(-\beta \Delta H)$ if $\Delta H > 0$.

It is easy to show that this dynamics is reversible with respect to the measure (2.2), hence, the unique invariant measure of the process is the equilibrium Gibbs measure. The problem we want to study is the way in which a system approaches the equilibrium state when it is prepared in the configuration with all the spins equal to minus one ($\sigma_0 = -\underline{1}$) and the magnetic field h is chosen positive but small with respect to the coupling constant J ($h/J < 1$), while β is very large.

In [NS1, RTMS, S1, TM1] this problem was studied in a box with periodic boundary conditions: it was shown that for β sufficiently large and h small enough, depending on the size of the box, the system shows metastable behavior. (The larger the box the larger β has to be.) This means that the system spends a long time $\tau_{p, \beta} \sim \exp(\beta(4J^2/h))$ in a phase with negative magnetization performing random wanderings near the configuration $-\underline{1}$. These wanderings are characterized by the formation of small droplets of pluses inside the sea of minuses which disappear quickly; their typical life time $\tau_{p, \beta}^s$ is much shorter than the lifetime of the metastable state ($\tau_{p, \beta}^s \ll \tau_{p, \beta}$).

A droplet of pluses will tend to grow however if it is large enough, i.e. when its dimension is larger than a critical length, $l^* = [2J/h] + 1$, where $[a]$ is the integer part of the real number a . The exit from the metastable phase is achieved, then, when a sufficiently large droplet shows up somewhere in the lattice: this droplet is called protocritical and in the limit $\beta \rightarrow \infty$ it is a square droplet with sides l^* . When this protocritical droplet appears, it grows and covers the whole lattice in a time $\tau_{p, \beta}^g$ which is very

small compared to the lifetime of the metastable phase. It has been shown that $\tau_{p,\beta}^s \ll \tau_{p,\beta}^g \ll \tau_{p,\beta}$.

In our case of free boundary conditions we show, rigorously in the limit $\beta \rightarrow \infty$ and via Monte Carlo simulations for β large, that the system exhibits metastability and that the lifetime of this metastable phase is $\tau_\beta \sim \exp(\beta(J^2/h))$. This exit time is much smaller than in the case of periodic boundary conditions. This is a consequence of the fact that the tendency of a droplet to grow is favoured when such a droplet has one of its sides on the boundary of the domain or at a distance one from it. Indeed, we have to introduce two different critical lengths, λ_1 and λ_2 , which refer respectively to droplets close to any part and far from the boundary. In the limit $\beta \rightarrow \infty$ we find

$$\lambda_1 = \left[\frac{J}{h} \right] + 1 \quad \text{and} \quad \lambda_2 = \left[\frac{2J}{h} \right] + 1 \quad (2.3)$$

with J and h fixed such that $J/h > 1$.

Finally, we remark that the exit from the metastable phase occurs through a critical seed which appears in one of the four corners of the lattice. It grows in a time $\tau_\beta \sim \exp(\beta(J-h))$ to cover the whole domain. Hence, in this case the position of the nucleation seed in the lattice can be predicted a priori. In [RKLRLN] it was observed that in the case of semi-periodic boundary conditions the critical droplet can show up on one of the two sides where the periodic boundary conditions are not imposed. This decay mode was previously noted by Tomita and Miyashita [TM2]. In [Sh] it has been shown that this phenomenon is responsible for the peculiar behavior of the large deviations probability in the case of free boundary conditions.

3. NUMERICAL RESULTS

In this section we describe some simulation results obtained using the Metropolis algorithm for an M by M square at low temperature and small magnetic fields; the typical values which have been used in the numerical experiments are $\beta \geq 2$, $h \leq 0.5$, and $J = 1$. This range of parameters is different from those considered, for instance, in [RTMS, TM1] and references therein, e.g. in [RTMS] they considered the case $J = 1$, $\beta = 1.102$ and h varying approximately in the range $0.04 \leq h \leq 0.9$.

To obtain a numerical estimate of the critical lengths λ_1, λ_2 we fixed β and h and prepared the system in the configuration $\sigma = -\underline{1}$ except for a plus droplet of size l in the corner of the square; the size of the square was

chosen $M = l + 5$. We considered decreasing values of l and in each experiment we found the smallest value of l such that the droplet grew. We took this as an estimate of the critical length λ_1 . The length λ_2 was measured in a similar way; the droplet now being placed at the center of the lattice.

Figure 1 presents an average over 60 independent determinations of λ_1 and λ_2 with inverse temperature $\beta = 10$, and $\beta = 6$ respectively. The solid and dashed lines represent the theoretical values (2.3), valid in the limit $\beta \rightarrow \infty$, while the black circles and black squares are, respectively, the numerical estimates of the critical lengths λ_1 and λ_2 . The error bars have been omitted because the statistical errors, evaluated as the empirical standard deviation over the square root of the number of experiments, were found to be very small. The agreement between the numerical result and the theoretical prediction is very good although the former do not show the staircase structure very clearly—due no doubt to the fact that β , while large, is not infinite. Low temperature calculations in the case of periodic boundary conditions were performed in [N1, LNR, N2].

We remark that the number of Monte Carlo steps per site (MCS) one has to wait in order to see the growth of the droplet greatly increases as the value of the magnetic field h is decreased; indeed this time, for β large,

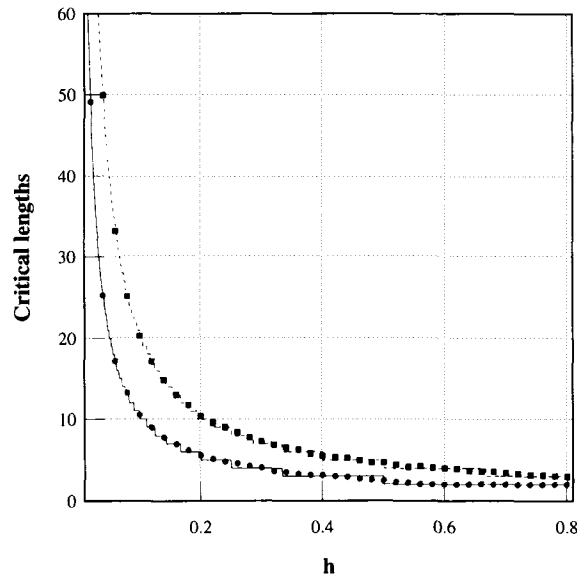


Fig. 1. The critical lengths λ_1 and λ_2 as functions of the magnetic field h . The solid and dashed lines represent the theoretical prediction (2.3) in the limit $\beta \rightarrow \infty$; the black circles and the black squares are, respectively, the numerical estimates of λ_1 at $\beta = 10$ and λ_2 at $\beta = 6$.

is approximately given by $\exp \beta(J-h)$ and $\exp \beta(2J-h)$ respectively in the case of a droplet close to or far from the boundary of the domain. The smallest magnetic field we have considered is $h=0.02$: in this case we have set $M=56$ and the number of MCS needed to see the shrinking or growth of the droplet was approximately 10^5 .

In Fig. 2 we have plotted the magnetization per spin of the whole box m_0 and the magnetization per spin m_1, m_2, m_3 and m_4 evaluated in four square boxes of side λ_1 placed at the four corners of the lattice, as functions of the number of iterations (time) in a single history of the system obtained after preparing the system in the starting configuration -1 . In both pictures the solid line represents the magnetization of the whole lattice m_0 , while the other lines refer to m_1, m_2, m_3 and m_4 . The top graph in Fig. 2

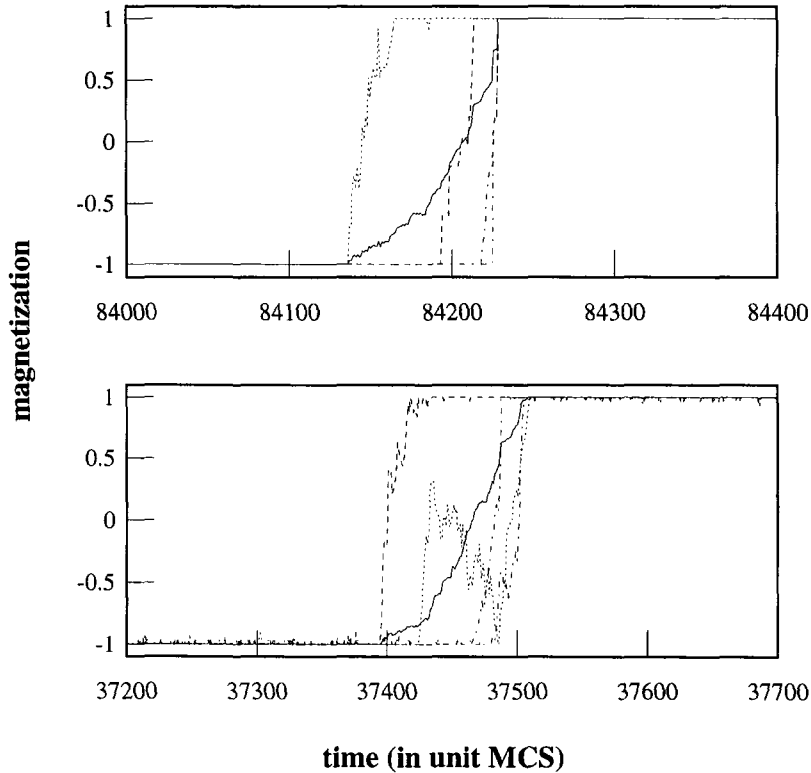


Fig. 2. The solid line represents the average magnetization of the whole lattice m_0 ; the other lines represent m_1, m_2, m_3 and m_4 . The top figure corresponds to the case $\beta=3$, $h=0.24$ ($\lambda_1=5$) and $M=16$; the bottom one to $\beta=2$, $h=0.14$ ($\lambda_1=8$) and $M=32$.

refers to the case where $\beta = 3$, $h = 0.24$ ($\lambda_1 = 5$) and $M = 16$; the one below has been obtained with $\beta = 2$, $h = 0.14$ ($\lambda_1 = 8$) and $M = 32$.

In both cases it is clear that the system stays for a long time (about 10^4 MCS) in the configuration -1 : the thermal fluctuations, visible for $\beta = 2$, are negligible for $\beta = 3$. After this long time the magnetization in one of the corner boxes flips to one (nucleation of the protocritical droplet); once this rare event has happened, all the other magnetizations start to grow and quickly reach the value $+1$; that is the system quickly reaches the equilibrium state.

Finally, we have evaluated the lifetime of the metastable state. In [RTMS, MT] and references therein the lifetime of the metastable state has been estimated at temperature $T = 0.8T_c$, where T_c is the Onsager critical temperature, corresponding to our $\beta = 1.102$. As the magnetic field is varied, four different regimes are detected (see [RTMS] for a detailed description of these different regimes); at low values of the magnetic field they find that the logarithm of the lifetime of the metastable state is a linear function of the inverse of the magnetic field. They call this the “single-droplet region,” meaning that the nucleation of the stable state is achieved via the formation of a single critical droplet, see also [RKLRN, TM2].

In Fig. 3 we have plotted our numerical measurements of the escape time τ_β versus the inverse of the magnetic field. It is clear that when the temperature is decreased the numerical results approach the theoretical value

$$\frac{1}{\beta} \log \tau_\beta \approx \frac{1}{h} \quad (3.1)$$

which by Theorem 1.1 is valid in the limit of zero temperature. The numerical data have been fitted with a linear function $1/\beta \log \tau_\beta = m_\beta 1/h + n_\beta$; the values of m_β and n_β are listed in Table 1. It is clear that the trend with

Table 1. Results of the Linear Fit
 $(1/\beta) \log \tau_\beta = m_\beta (1/h) + n_\beta$ of the Data
in Fig. 3, for $\beta \rightarrow \infty$, $m_\beta \rightarrow 1$, $n_\beta \rightarrow \infty$

β	m_β	n_β
1.00	0.216	1.816
1.25	0.312	1.173
1.50	0.483	0.669
1.75	0.629	0.378
2.00	0.782	0.083

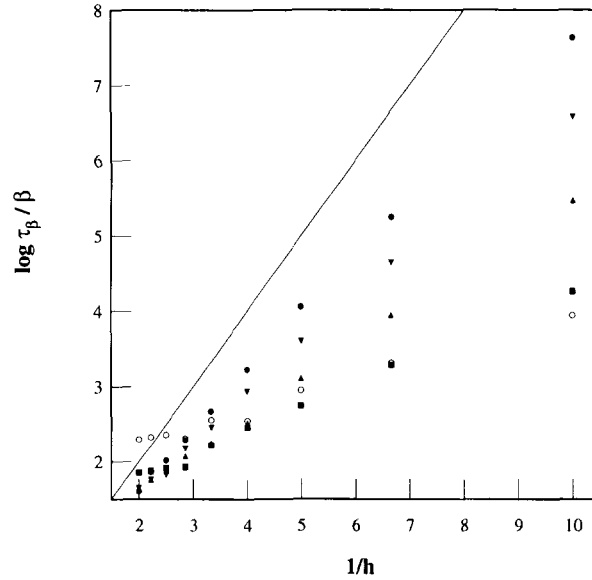


Fig. 3. Numerical estimates of $\log \tau_\beta / \beta$ plotted versus $1/h$ for different values of the inverse temperature β . Empty circles, black squares, black upward triangles, black downward triangles and black circles refer respectively to $\beta = 1.00, 1.25, 1.50, 1.75, 2.00$. All the results are averages over 60 different histories for $M = 32$. The solid line is the graph of a linear function with slope equal to 1.

increasing β is correct, although the value (3.1) would be reached only for much larger values of β . Such simulation would require very long computer time.

We remark that (3.1) is valid in the limit of zero temperature, while in the case of finite temperature it is only an approximation of the correct behavior $\tau_\beta = C(\beta) h^{-3} \exp\{\beta F_c(\beta, h)\}$ where $C(\beta)$ is a model dependent function of the temperature and $F_c(\beta, h)$ is the free energy cost of a critical droplet [N1, N2, LNR, RG]. Fitting our data by means of this expression does not give a good result; in order to improve this fit we should consider smaller values of the external field.

The results in Fig. 3 have been obtained in the case $M = 32$ and are the average of 60 different histories. We have checked that very similar results are obtained if one considers larger domains (for instance $M = 64, 128$) but have not performed extensive statistics in these situations because the behavior of τ_β at small values of the magnetic field and very low temperatures does not depend on the size of the lattice (this is confirmed by the results in [RTMS], see Fig. 2 there). Finally, we remark that the linearity of the logarithm of τ_β is lost when $1/h$ is small enough, this is because for h large enough the system is in the multi-droplet regime.

4. RIGOROUS RESULTS

We list some definitions and notations which are necessary to discuss our rigorous results:

1. The spin configurations $\sigma, \eta \in \Omega$, are called *nearest neighbor* configurations iff $\exists x \in A$ such that $\eta = \sigma^x$, where σ^x is the configuration obtained by flipping in σ the spin at site x . A *path* is a sequence of configurations $\sigma_0 \sigma_1 \cdots \sigma_n$ such that for $i = 1, \dots, n$, σ_{i-1} and σ_i are nearest neighbors. A path $\sigma_0 \sigma_1 \cdots \sigma_n$ is called *downhill* iff $H(\sigma_{i+1}) \leq H(\sigma_i)$ for $i = 0, 1, \dots, n-1$.

2. Given $A \subset \Omega, \eta \in \Omega$ we define the *hitting* or *first passage time*

$$\tau_A^\eta = \inf \{ t \geq 0 : \sigma_t^\eta \in A \} \quad (4.1)$$

where σ_t^η is the stochastic process starting from the configuration η .

3. A *local minimum* of the Hamiltonian is a configuration σ such that one has

$$H(\sigma^x) > H(\sigma) \quad \forall x \in A \quad (4.2)$$

A local minimum will also be called a *stable configuration* because starting from it the system will not move for a time which is exponentially long in the inverse temperature β .

4. The set of all the local minima is denoted by $\mathcal{M} \subset \Omega$.

5. Given $\eta \in \mathcal{M}$, we consider the process starting from η and say

$$\begin{aligned} \eta \text{ subcritical} &\Leftrightarrow \lim_{\beta \rightarrow \infty} P(\tau_{-1}^\eta < \tau_{+1}^\eta) = 1 \\ \eta \text{ supercritical} &\Leftrightarrow \lim_{\beta \rightarrow \infty} P(\tau_{+1}^\eta < \tau_{-1}^\eta) = 1 \end{aligned} \quad (4.3)$$

where $P(X)$ is the probability of the event X .

6. Given $\sigma \in \mathcal{M}$ we define its *basin of attraction*

$$B(\sigma) = \{ \eta \in \Omega : \text{all downhill paths starting from } \eta \text{ end in } \sigma \} \quad (4.5)$$

We observe that since h is the smallest possible difference of energy between two nearest neighbor configurations, hence after

$$\mathcal{F} = \frac{\max_{\eta \in \Omega} H(\eta) - \min_{\eta \in \Omega} H(\eta)}{h}$$

steps a downhill path must necessarily end in a local minimum.

- 7. Given $\mathcal{G} \subset \Omega$, \mathcal{G} is *connected* iff $\forall \eta, \sigma \in \mathcal{G} \exists$ a path $\omega \subset \mathcal{G}$ starting from σ and ending in η ; we will say that this path *connects* σ to η .
- 8. Given a connected set $\mathcal{G} \subset \Omega$, we will call the *boundary* of \mathcal{G} the set

$$\partial \mathcal{G} = \{ \eta \in \Omega: \eta \notin \mathcal{G}, \exists x \in \Lambda: \eta^x \in \mathcal{G} \} \tag{4.5}$$

- 9. Given two rectangles R_1 and R_2 on the dual lattice $\Lambda + (\frac{1}{2}, \frac{1}{2})$, we say that R_1 and R_2 are *interacting* rectangles iff R_1 and R_2 intersect or are separated by one lattice spacing. If two such rectangles have only two corners at distance one, then they are considered not interacting.

Let $h > 0$ and start the system from the configuration $\sigma_0 = -\underline{1}$. We will now describe how the stable configuration $+\underline{1}$ is approached and we will evaluate how long the system remains in the metastable phase. To do this we first characterize the local minima of the hamiltonian (2.1).

Lemma 1.1. Let us consider model (2.1) with $J > h > 0$ and $M > 2$. Then $\sigma \in \mathcal{M}$ iff $\sigma(x) = -1 \forall x \in \Lambda$ except for sites which are inside some rectangles R_1, \dots, R_n laying on the dual lattice $\Lambda + (\frac{1}{2}, \frac{1}{2})$ such that $\forall i, j = 1, \dots, n$ and $i \neq j$

- (i) R_i and R_j are not interacting
- (ii) R_i has sides of length greater than two
- (iii) R_i cannot have one of its sides on the “border” of Λ and one of its two other sides perpendicular to this one at a distance one from the border (see Fig. 4).

Remarks. 1. We denote by $\mathcal{R}(l_1, l_2)$ with $2 \leq l_1, l_2 \leq M$ the set of all configurations with all spins -1 except for those inside a rectangle with sides l_1 and l_2 and such that they are local minima.

- 2. Given a local minimum $\sigma \in \mathcal{R}(l_1, l_2)$ we denote it by $R_{l,m}$ where

$$l = \min\{l_1, l_2\} \quad \text{and} \quad m = \max\{l_1, l_2\} \tag{4.6}$$

- 3. We denote by $\mathcal{R} \subset \mathcal{M}$ the set of all local minima containing only one rectangle of pluses.

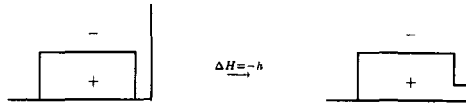


Fig. 4. Example of a rectangular droplet which is not a local minimum.

Now, we state under which conditions a local minimum is subcritical, that is we describe the evolution of the system starting from a stable configuration. Consider a local minimum $R_{l,m} \in \mathcal{R}$:

Lemma 1.2. When each side of $R_{l,m}$ is at least at distance two from the border of the lattice, then given $\varepsilon > 0$ one has

(i) $l < \lambda_2 \Rightarrow R_{l,m}$ is subcritical and

$$P(e^{\beta(l-1)h - \beta\varepsilon} < \tau_{-1}^{R_{l,m}} < e^{\beta(l-1)h + \beta\varepsilon}) \xrightarrow{\beta \rightarrow \infty} 1 \quad (4.7)$$

(ii) $l \geq \lambda_2 \Rightarrow R_{l,m}$ is supercritical and

$$P(e^{\beta(2J-h) - \beta\varepsilon} < \tau_{+1}^{R_{l,m}} < e^{\beta(2J-h) + \beta\varepsilon}) \xrightarrow{\beta \rightarrow \infty} 1 \quad (4.8)$$

When at least one of the sides of $R_{l,m}$ is at distance one from the border of the square or is laying on it, then given $\varepsilon > 0$ one has

(i) $l < \lambda_1, m \leq M-1 \Rightarrow R_{l,m}$ is subcritical and

$$P(e^{\beta(l-1)h - \beta\varepsilon} < \tau_{-1}^{R_{l,m}} < e^{\beta(l-1)h + \beta\varepsilon}) \xrightarrow{\beta \rightarrow \infty} 1 \quad (4.9)$$

(ii) $l \geq \lambda_1 \Rightarrow R_{l,m}$ is supercritical and

$$P(e^{\beta(J-h) - \beta\varepsilon} < \tau_{+1}^{R_{l,m}} < e^{\beta(J-h) + \beta\varepsilon}) \xrightarrow{\beta \rightarrow \infty} 1 \quad (4.10)$$

Remarks. 1. In the second case, if $m = M$ then the local minimum is supercritical, no matter how long the smallest side l is.

2. If one considers a local minimum with more than one rectangle, then this local minimum is subcritical iff all its rectangles are “subcritical” (abuse of language).

3. It is possible to prove a stronger version of Lemma 1.2 that gives rise to a more detailed description of the contraction or the growth of the droplet (see Lemma 3 and 4 in [KO1], Theorem 1 in [NS1]).

The proof of this lemma is the standard one, see, for instance, [NS1, KO1, KO2] or the proof of Proposition 4.1 in [CO]. One has to consider the basin of attraction of the local minimum $R_{l,m}$ and work out the minimum of the energy on its boundary. Once this has been done, everything follows via the general arguments in Proposition 3.7 in [OS1]. The difference with respect to the case of periodic boundary conditions is that one must take into account situations like the one depicted in Fig. 5, that are absent in the Ising model with periodic boundary conditions.

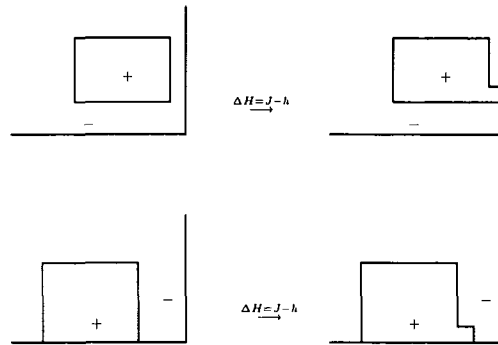


Fig. 5. Mechanisms of growth peculiar to the Ising model with free boundary conditions.

We can formulate, now, the theorem which describes the exit from the metastable phase: this theorem states that with high probability the system will visit a particular configuration \mathcal{P} before reaching $+\underline{1}$ and that the exit time is dominated by the time the system needs to reach \mathcal{P} .

This *protocritical* configuration \mathcal{P} is such that all the spins are minuses except those in the union of a $\lambda_1 \times (\lambda_1 - 1)$ rectangle, one of whose corners coincides with one of the corners of the domain, and a unit square laying on the border of Λ and touching the longer side of the rectangle (see Fig. 6).

Setting

$$\Gamma = H(\mathcal{P}) - H(-\underline{1}) = 2J\lambda_1 - h(\lambda_1^2 - \lambda_1 + 1) \tag{4.11}$$

we consider the process σ_t starting from $-\underline{1}$ and define $\bar{\tau}_{-1}$ as the last time the configuration was $-\underline{1}$ before it became $+\underline{1}$,

$$\bar{\tau}_{-1} = \sup\{t < \tau_{+1} : \sigma_t = -\underline{1}\} \tag{4.12}$$

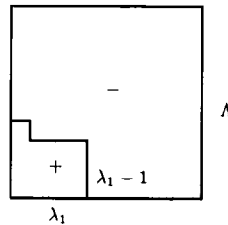


Fig. 6. Protocritical droplet \mathcal{P} .

Similarly we call $\bar{\tau}_{\mathcal{P}}$ the first time after $\bar{\tau}_{-1}$ when $\sigma_t = \mathcal{P}$,

$$\bar{\tau}_{\mathcal{P}} = \inf\{t > \bar{\tau}_{-1} : \sigma_t = \mathcal{P}\} \quad (4.13)$$

Finally, we can state the following theorem:

Theorem 1.1. Given $\varepsilon > 0$,

- (i) $P_{-1}(\bar{\tau}_{\mathcal{P}} < \tau_{+1}) \xrightarrow{\beta \rightarrow \infty} 1$
- (ii) $P_{-1}(e^{\beta I - \beta\varepsilon} < \tau_{+1} < e^{\beta I + \beta\varepsilon}) \xrightarrow{\beta \rightarrow \infty} 1$

N.B. Theorem 1.1 is for a fixed A ; the convergence to one is not uniform in A .

From Lemma 1.2 and Theorem 1.1 we have a rather accurate description of the system in the metastable phase: starting from -1 the system will spend a lot of time “close” to this configuration; sometimes small droplet of pluses appear, but the system quickly goes back to -1 . Only after a long time, compared to the time these fluctuations need, the system will nucleate the protocritical droplet \mathcal{P} and it will then reach the stable phase in a relatively short time.

It is possible to give a more detailed description of the first excursion from -1 to $+1$; one could state a result similar to the one in [S1] (see also Theorem 3 in [KO1]). We do not enter in the details of this construction for the case of free boundary conditions, because no relevant difference appears with respect to the case of periodic boundary conditions. Roughly speaking one can say that the system, during the excursion from -1 to $+1$, will follow a rather well-defined sequence of configurations made up of growing rectangular, almost square, droplets located at one of the four corners of A .

The proof of Theorem 1.1 is now sketched: as in [KO1] we can define a set \mathcal{A} satisfying some properties which will be listed below. The construction of the set \mathcal{A} is exactly as in [KO1], except for the fact that whenever there is a rectangle with one side on the “border” and another side at distance one from the border of A , the rectangle is enlarged so that the latter side also touches the border (see e.g. Fig. 4).

The relevant properties of the set \mathcal{A} are the following:

- (i) \mathcal{A} is connected; $-1 \in \mathcal{A}$ and $+1 \notin \mathcal{A}$.
- (ii) There exists a path ω connecting -1 with \mathcal{P} contained in \mathcal{A} such that

$$H(\sigma) < H(\mathcal{P}) \quad \forall \sigma \in \omega, \quad \sigma \neq \mathcal{P}$$

There exists a path ω' connecting \mathcal{P} with $+\underline{1}$ contained in \mathcal{A}^c such that

$$H(\sigma) < H(\mathcal{P}) \quad \forall \sigma \in \omega', \quad \sigma \neq \mathcal{P}$$

(iii) The minimal energy in $\partial\mathcal{A}$ is attained for the “protocritical” configuration; namely

$$\min_{\sigma \in \partial\mathcal{A}} (H(\sigma) - H(-\underline{1})) = H(\mathcal{P}) - H(-\underline{1}) = \Gamma \quad (4.14)$$

and

$$\min_{\sigma \in \partial\mathcal{A} \setminus \{\mathcal{P}\}} (H(\sigma) - H(\mathcal{P})) > 0 \quad (4.15)$$

(iv) With probability greater than zero, uniformly in β , the system starting from \mathcal{P} will reach $+\underline{1}$ before visiting $-\underline{1}$; namely, given $\varepsilon > 0$

$$P(\tau_{+\underline{1}}^{\mathcal{P}} < \tau_{-\underline{1}}^{\mathcal{P}}) \geq e^{-\varepsilon\beta}$$

and

$$P(\tau_{+\underline{1}}^{\mathcal{P}} < e^{\beta(J-h) + \beta\varepsilon} | \tau_{+\underline{1}}^{\mathcal{P}} < \tau_{-\underline{1}}^{\mathcal{P}}) \xrightarrow{\beta \rightarrow \infty} 1$$

Using properties (i)–(iv) and Propositions 3.4, 3.7 in [OS1] we get Theorem 1.1.

5. CONCLUDING REMARKS

In this paper we have studied metastability in the two-dimensional Ising model on an M by M square with free boundary conditions rigorously in the limit $\beta \rightarrow \infty$ and via Monte Carlo simulations at finite temperatures. We found good agreement between the theoretical predictions and the simulations and for a large range of h and low enough temperatures. The qualitative agreement persists even above one half of T_c .

Compared to periodic boundary conditions there are two relevant differences: (i) the critical length of the droplet and hence the life time of the metastable phase is much shorter; (ii) the protocritical droplet is always at one of the four corners of the square.

It is clear that our analysis applies equally well to a rectangular domain with sufficiently long sides. In fact the basic approach carries over, in principle, to a general domain with general boundary conditions as long as there is no wetting: see [RKLRN] for the wetting case. The protocritical domain will always form, when $\beta \rightarrow \infty$ at the place (or places) where the energy cost, $H(\mathcal{P}) - H(\sigma_{-1})$, is minimal.

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